Measure Algebras

Thomas Jech

Hejnice, February 2016

The result

These notes contain the complete proof of the following result:

Theorem. A Boolean σ-algebra B is a measure algebra if and only if it is weakly distributive and uniformly concentrated.

Boolean σ-algebras

A *Boolean algebra* is an algebra B of subsets of a given nonempty set S, with Boolean operations $a \cup b$, $a \cap b$, $-a = S - a$, and the zero and unit elements $\mathbf{0} = \emptyset$ and $\mathbf{1} = S$.

A *Boolean σ-algebra* is a Boolean algebra B such that every countable set $A \subset B$ has a supremum $\sup A = \bigvee A$ (and an infimum $\inf A = \bigwedge A$) in the partial ordering of B by inclusion.

Let B be a Boolean algebra and let $B^+ = B - \{\mathbf{0}\}$. A set $A \subset B^+$ is an *antichain* if $a \cap b = \mathbf{0}$ whenever a and b are distinct elements of A. A *partition* W (of $\mathbf{1}$) is a maximal antichain, i.e. an antichain with $\bigvee W = \mathbf{1}$. B satisfies the *countable chain condition* (ccc) if it has no uncountable antichains.

A *measure* (a strictly positive σ-additive probability measure) on a Boolean σ-algebra B is a real valued function m on B such that

(i) $m(\mathbf{0}) = 0$, $m(a) > 0$ for $a \neq \mathbf{0}$, and $m(\mathbf{1}) = 1$

(ii) $m(a) \leq m(b)$ if $a \subset b$

(iii) $m(a \cup b) = m(a) + m(b)$ if $a \cap b = \mathbf{0}$

(iv) $m(\bigvee_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} m(a_n)$ if the a_n are pairwise disjoint.

A *measure algebra* is a Boolean σ-algebra that carries a measure.

A function m that satisfies (i), (ii) and (iii) above is called a *(strictly positive) finitely additive measure*. And m satisfies (iv) if and only if it is *continuous*:

(iv) if $\{a_n\}_n$ is a decreasing sequence in B with $\bigwedge_{n=1}^{\infty} a_n = \mathbf{0}$ then $\lim_n m(a_n) = \mathbf{0}$.

Every measure algebra is ccc because $B^+ = \bigcup_{n=1}^{\infty} \{C_n\}_n$ where $C_n = \{a : m(a) \geq 1/n\}$ and every antichain in C_n has size $\leq n$.

Convergence

If $\{a_n\}_n$ is a sequence in a Boolean σ-algebra B, one defines

$$\limsup_n a_n = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} a_k \quad \text{and} \quad \liminf_n a_n = \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} a_k,$$

and if $\limsup_n a_n = \liminf_n a_n = a$, then a is the *limit* of the sequence, denoted $\lim_n a_n$.

A sequence $\{a_n\}_n$ converges to $\mathbf{0}$ if and only if $\limsup a_n = \mathbf{0}$ if and only if there exist $b_n \geq a_n$, b_n decreasing, with $\bigwedge_n b_n = \mathbf{0}$. A sequence $\{a_n\}_n$ converges to $\mathbf{1}$ if and only if $\liminf a_n = \mathbf{1}$. (Exercise: If $\lim a_n = \lim b_n = \mathbf{0}$ then $\lim(a_n \cup b_n) = \mathbf{0}$).

If $\{a_n\}_n$ is an antichain then $\lim a_n = \mathbf{0}$.

In a measure algebra, if $\lim a_n = \mathbf{0}$ then $\lim m(a_n) = 0$. If $\sum m(a_n) < \infty$ then $\lim a_n = \mathbf{0}$.

An algebraic characterization of measure algebras

A Boolean σ-algebra B is *weakly distributive* if whenever $\{W_n\}_n$ is a sequence of countable maximal antichains then each W_n has a finite subset E_n such that $\lim_n \bigcup E_n = \mathbf{1}$.

Equivalently, if for every k, $\{a_{kn}\}_n$ is an increasing sequence with $\bigvee_n a_{kn} = \mathbf{1}$ then there exists a function $f : N \to N$ such that $\lim a_{k,f(k)} = \mathbf{1}$.

Or, if for every k, $\{a_{kn}\}_n$ is a decreasing sequence with $\bigwedge_n a_{kn} = \mathbf{0}$ then there exists a function $f : N \to N$ such that $\lim a_{k,f(k)} = \mathbf{0}$.

A Boolean σ-algebra B is *uniformly weakly distributive* if there exists a sequence of functions $\{F_n\}_n$ such that for each countable maximal antichain W, $F_n(W)$ is a finite subset of W, and if $\{W_n\}_n$ is a sequence of countable maximal antichains then $\lim_n \bigcup F_n(W_n) = \mathbf{1}$.

If B is a measure algebra then B is uniformly weakly distributive: For every n, if W is a countable maximal antichain, let $F_n(W) = E \subset W$ be such that $m(\bigcup E) \geq 1 - 1/2^n$.

An algebraic characterization of measure algebras

A Boolean σ-algebra B is *concentrated* if for every sequence $\{A_n\}$ of finite antichains with $|A_n| \geq 2^n$ there exist $a_n \in A_n$ such that $\lim a_n = \mathbf{0}$.

B is *uniformly concentrated* if there exists a function F such that for each finite antichain A, $F(A)$ is an element of A, and if A_n is a sequence of finite antichains with $|A_n| \geq 2^n$ then $\lim_n F(A_n) = \mathbf{0}$.

A measure algebra is uniformly concentrated: if A is a finite antichain, let $F(A)$ be an element of A of least measure (then $m(F(A)) \leq 1/|A|$.)

Theorem. A Boolean σ-algebra B is a measure algebra if and only if it is weakly distributive and uniformly concentrated.

Fragmentations

A *fragmentation* of B is a sequence of subsets $C_1 \subset C_2 \subset \ldots \subset C_n \subset \ldots$ such that $\bigcup_n C_n = B^+$ and for every n, if $a \in C_n$ and $a \leq b$ then $b \in C_n$.

A fragmentation is *σ-finite cc* if for every n, every antichain $A \subset C_n$ is finite.

A fragmentation is *σ-bounded cc* if for every n there is a constant K_n such that every antichain $A \subset C_n$ has size $\leq K_n$.

A fragmentation is G_δ if for every n, no sequence in C_n converges to $\mathbf{0}$.

A fragmentation is *tight* if whenever $\{a_n\}_n$ is a sequence such that $a_n \notin C_n$ for every n, then $\lim a_n = \mathbf{0}$.

A fragmentation is *graded* if for every n, whenever $a \cup b \in C_n$ then either $a \in C_{n+1}$ or $b \in C_{n+1}$.

Fragmentations

The name G_δ comes from the fact that if B is weakly distributive and the fragmentation is G_δ then the set $\{\mathbf{0}\}$ is a G_δ set in the convergence topology.
A G_δ fragmentation is σ-finite cc.

In a measure algebra, the fragmentation defined by
$C_n = \{a : m(a) \geq 1/2^n\}$ has all the above properties.

Lemma. (a) If B is uniformly concentrated then B has a tight σ-bounded cc fragmentation.
(b) If B is weakly distributive and has a tight σ-finite cc fragmentation then B is uniformly weakly distributive.

Consequently, a weakly distributive, uniformly concentrated σ-algebra is uniformly weakly distributive.

Proof. (a) Let F be a function on finite antichains that witnesses that B is uniformly concentrated. For every n let C_n be the set of all $a \neq \mathbf{0}$ such that there is no antichain A with $|A| \geq 2^n$ and $a \leq F(A)$.

Fragmentations

It is easy to see that C_n is a fragmentation: If $a \notin \bigcup_n C_n$ then for every n there exists an antichain A_n with $|A_n| \geq 2^n$ and $a \leq F(A_n)$. Hence $a \leq \lim F(A_n) = \mathbf{0}$, and so $a = \mathbf{0}$.

The fragmentation is tight because if $a_n \notin C_n$ then $a_n \leq F(A_n)$ for some A_n and so $\lim a_n = \mathbf{0}$. It is σ-bounded cc because if $A \subset C_n$ is an antichain then $F(A) \in C_n$ and so $|A| < 2^n$.

(b) Let B be weakly distributive and let $\{C_n\}_n$ be a tight σ-finite cc fragmentation; we shall find the functions F_n witnessing that B is uniformly weakly distributive.

Let $n \in N$, and let W be a countable maximal antichain. We claim that there exists a finite set $E \subset W$ such that for every finite $E' \subset W - E$, $\bigcup E' \notin C_n$: otherwise we find an infinite sequence $\{E_k\}_k$ of pairwise disjoint nonempty subsets of W with $\bigcup E_k \in C_n$, an infinite antichain in C_n. We let $F_n(W)$ be this E.

Now let W_n, $n \in N$, be countable maximal antichains. We claim that $\lim \bigcup F_n(W_n) = \mathbf{1}$. Since B is weakly distributive, there exist finite sets $E_n \subset W_n$ such that $\lim \bigcup E_n = \mathbf{1}$. For each n let $a_n = \bigcup E_n - \bigcup F_n(W_n)$. By the claim above, $a_n \notin C_n$. Because $\{C_n\}_n$ is tight, we have $\lim a_n = \mathbf{0}$, and because $\bigcup F_n(W_n) \geq \bigcup E_n \cap (-a_n)$, we have $\lim \bigcup F_n(W_n) = \lim \bigcup E_n = \lim -a_n = \mathbf{1}$.

Lemma. A Boolean $\sigma-$algebra is uniformly weakly distributive if and only if it has a tight G_δ fragmentation.

Proof. First assume that B is uniformly weakly distributive and let F_n be functions witnessing it. For each n we let C_n be the set of all a such that for some $k \leq n$ $a \cap \bigcup F_k(W) \neq \mathbf{0}$ for every countable maximal antichain W.

Fragmentations

To show that $\{C_n\}_n$ is a fragmentation, we show that $\bigcup_n C_n = B^+$: if $a \notin C_n$ for all n then for all k there is a W_k such that $a \cap \bigcup F_k(W_k) = \mathbf{0}$, and because $\lim(a \cap \bigcup F_k(W_k) = a$ we have $a = \mathbf{0}$.

To show that $\{C_n\}_n$ is tight, let $a_n \notin C_n$ for each n. For each n there is a W_n such that $a_n \cap b_n = \mathbf{0}$ where $b_n = \bigcup F_n(W_n)$. Since $\lim b_n = \mathbf{1}$, we have $\lim -b_n = \mathbf{0}$, and because $a_n \leq -b_n$, it follows that $\lim a_n = \mathbf{0}$.

To show that $\{C_n\}_n$ is G_δ, let $n \in N$ and let $\lim a_k = \mathbf{0}$; it suffices to find a $k \in N$ such that $a_k \notin C_n$. We may assume that $\{a_k\}_k$ is strictly decreasing and let W be the maximal antichain $\{a_{k-1} - a_k\}_k$ where $a_0 = \mathbf{1}$. Let $E = F_1(W) \cup ... \cup F_n(W)$. There exists a k large enough so that $a_k \cap \bigcup E = \mathbf{0}$. It follows that $a_k \notin C_n$.

Fragmentations

For the converse, let $\{C_n\}_n$ be a tight G_δ fragmentation. In view of the preceding lemma it suffices to show that B is weakly distributive. For every k, let $\{a_{kn}\}_n$ be a decreasing sequence with $\bigwedge_n a_{kn} = \mathbf{0}$. We shall find a function $f : N \to N$ such that $\lim a_{k,f(k)} = \mathbf{0}$. Given $k \in N$, there is some $f(k)$ such that $a_{k,f(k)} \notin C_k$, as $\{C_n\}_n$ is G_δ. Because $\{C_n\}_n$ is tight, $\lim a_{k,f(k)} = \mathbf{0}$ follows.

A tight G_δ fragmentation is essentially unique: if C_n and C'_n are such, then for each n there is a k such that $C_n \subset C'_k$. (If not, there exists an n such that for all k there is some $a_k \in C_n - C'_k$ and so $\lim a_k = \mathbf{0}$).
(In the appendix we show that B is uniformly weakly distributive if and only if B is a Maharam algebra.)

Lemma. If $\{C_n\}_n$ is a G_δ fragmentation of B and if B is concentrated then $\{C_n\}_n$ is $\sigma-$bounded cc.

Proof. By contradiction, assume that for some n, there exist arbitrarily large finite antichains in C_n, and for each k, let A_k be an antichain in C_n of size $\geq 2^k$. Since B is concentrated, there exist $a_k \in A_k$ with $\lim a_k = \mathbf{0}$, a contradiction.

Lemma. If $\{C_n\}_n$ is a tight G_δ fragmentation then for every n there exists a $k > n$ such that for every $c \in C_n$, if $c = a \cup b$ then either $a \in C_k$ or $b \in C_k$.
Thus if B has a tight G_δ fragmentation then B has one that is also graded (i.e. $a \cup b \in C_n$ implies that either $a \in C_{n+1}$ or $b \in C_{n+1}$.)

Proof. Otherwise, for every $k > n$ there exist $c_k = a_k \cup b_k \in C_n$ such that $a_k \notin C_k$ and $b_k \notin C_k$. By tightness, $\lim a_k = \lim b_k = \mathbf{0}$ and so $\lim c_k = \mathbf{0}$, a contradiction.

In conclusion, we proved that a weakly distributive uniformly concentrated Boolean $\sigma-$algebra has a graded $\sigma-$bounded cc fragmentation.

In this lecture we introduce Kelley's condition for the existence of finitely additive measure on a Boolean algebra. But first we show how the measure problem reduces to finitely additive measures.

Theorem (Pinsker, Kelley). A Boolean σ-algebra B carries a measure if an only if it is weakly distributive and carries a finitely additive measure.

Proof. Let m be a finitely additive measure on B. We let

$$\mu(b) = \inf\{\lim_n m(b \cap u_n)\}$$

where the infimum is taken over all increasing sequences $\{u_n\}_n$ with $\bigvee_n u_n = \mathbf{1}$.
We show that μ is a σ-additive measure, and if B is weakly distributive then μ is strictly positive.

First show that $\mu(a \cup b) = \mu(a) + \mu(b)$ if $a \cap b = \mathbf{0}$. If $s = \{u_n\}_n$, let $\mu_s(x) = \lim_n m(x \cap u_n)$. Clearly, $\mu_s(a \cup b) = \mu_s(a) + \mu_s(b)$, and so $\mu(a) + \mu(b) \leq \mu_s(a \cup b)$. Hence $\mu(a) + \mu(b) \leq \mu(a \cup b)$. For each $\varepsilon > 0$ there is a sequence $s = s_a = \{u_n\}_n$ such that $\mu_s(a) \leq \mu(a) + \varepsilon$, and similarly $s_b = \{v_n\}_n$. Let $s = \{u_n \cap v_n\}_n$. Then $\mu(a \cup b) \leq \mu_s(a \cup b) = \mu_s(a) + \mu_s(b) \leq \mu_{s_a}(a) + \mu_{s_b}(b) \leq \mu(a) + \mu(b) + 2\varepsilon$, and the equality follows.

To show the continuity of μ, let a_n be a decreasing sequence with $\bigwedge_n a_n = \mathbf{0}$; we show that $\lim \mu(a_n) = 0$. Let $\varepsilon > 0$.
Let $M = \lim m(a_n)$, and let K be such that $m(a_K) - M < \varepsilon$. Let $s = \{u_n\}_n$ where $u_n = -a_n$. As for all $k, n \geq K$, $m(a_k - a_n) < \varepsilon$, we have, for every $k \geq K$,
$\mu_s(a_k) = \lim_n m(a_k \cap u_n) = \lim_n m(a_k - a_n) \leq \varepsilon$, and hence $\mu(a_k) \leq \varepsilon$.

Kelley's Theorem

Finally, assume that B is weakly distributive, and let $b \in B$ be such that $\mu(b) = 0$; we show that $b = \mathbf{0}$. As $\mu(b) = 0$, there is for each k an increasing sequence $\{u_{kn}\}_n$ with $\bigvee_n u_{kn} = \mathbf{1}$ such that $\lim_n m(b \cap u_{kn}) < 1/k$.

By weak distributivity there is a function f such that $\lim u_{k,f(k)} = \mathbf{1}$. Hence $\bigvee_n \bigwedge_{k \geq n} u_{k,f(k)} = \mathbf{1}$. For each k let $a_k = b \cap u_{k,f(k)}$. We have $\bigvee_n \bigwedge_{k \geq n} a_k = b$. But because $m(a_k) < 1/k$, it follows that $\bigwedge_{k \geq n} a_k = \mathbf{0}$, and so $b = \bigvee_n \bigwedge_{k \geq n} a_k = \mathbf{0}$.

The intersection number

In order to state and prove Kelley's Theorem, we now work with Boolean set algebras, and use the term "finitely additive" for measures that are not necessarily strictly positive - when we need the condition $m(a) > 0$ if $a \neq \mathbf{0}$ we call m strictly positive.

Let B be a Boolean set algebra, $B \subset P(S)$ for some set S.
Let C be a subset of B^+. For every finite sequence $s = \langle c_1, ...c_n \rangle$ in C, let $\kappa_s = k/n$ where k is the largest size of a subset $J \subset \{1, ..., n\}$ such that $\bigcap_{i \in J} c_i$ is nonempty. The *intersection number* of C is the infimum $\kappa = \inf \kappa_s$ over all finite sequences s in C.
(The sequences s do not have to be nonrepeating.)

Note that for any n_0, the infimum $\inf \kappa_s$ taken over all sequences s of length $n \geq n_0$ is still κ: if s is a sequence of length $n < n_0$, let t be such that $t \cdot n \geq n_0$, and let s^* be a sequence we get when repeating each term of s t-times. Then $\kappa_{s^*} = \kappa_s$.

To better understand the significance of the intersection number, assume that m is a finitely additive measure on B, and let $C \subset B^+$. Let M be such that $m(c) \geq M$ for all $c \in C$. We show that the intersection number κ of C is at least M.
Let $s = \langle c_1, ...c_n \rangle$ be a sequence in C. For each $i \leq n$, let K_i be the characteristic function of c_i, i.e. $K_i(x) = 1$ if $x \in c_i$ and $= 0$ if $x \notin c_i$. Let $g = \sum_{i=1}^n K_i$ and consider $I_s = \int g \, dm$, the area below the graph of g. Since $\int K_i = m(c_i)$, we have
$I_s = \sum_i m(c_i) \geq M \cdot n$ and so (because $m(S) = 1$)
$||g|| = \max_{x \in S} g(x) \geq I_s \geq M \cdot n$.
Thus there exists some $x \in S$ such that $\sum_i K_i(x) \geq M \cdot n$; in other words, x belongs to at least $M \cdot n$ members of the sequence.
Hence $\kappa_s \geq M \cdot n/n = M$ and consequently $\kappa \geq M$.

Theorem (Kelley). Let $C \subset B^+$ have a positive intersection number κ. Then there exists a finitely additive measure m on B such that $m(c) \geq \kappa$ for all $c \in C$.

Corollary. If a Boolean algebra B has a fragmentation $\{C_n\}$ such that each C_n has a positive intersection number, then B carries a strictly positive finitely additive measure.

Proof of Corollary. For each n let m_n be positive on C_n. If we let $m(a) = \sum_n m_n(a)/2^n$, m is a strictly positive, finitely additive measure on B.

To prove Kelley's Theorem and construct a finitely additive measure on B we shall consider the vector space of all bounded functions on S (including all characteristic functions K_a for all $a \in B$) and find a linear functional F such that $F(\mathbf{1}) = 1$, $F(K_a) \geq 0$, and $F(K_c) \geq \kappa$ for all $c \in C$. Then we let $m(a) = F(K_a)$ for all $a \in B$.

To find the linear functional we use the **Hahn-Banach Theorem:**

Let p be a function such that $p(x) \geq 0$ for all x, $p(x+y) \leq p(x) + p(y)$, $p(\alpha\, x) = \alpha\, p(x)$ for all $\alpha \geq 0$, and $p(\mathbf{1}) \geq 1$.
Then there exists a linear functional F such that $F(\mathbf{1}) = 1$ and $F(x) \leq p(x)$ for all x.

Proof of Kelley's Theorem. Let $C \subset B^+ \subset P(S)$ and let κ be the intersection number of C. Let V be the vector space of all bounded functions on S with the supremum norm $||f|| = \sup\{|f(x)| : x \in S\}$. We shall find a linear functional F on V such that $0 \leq F(K_a) \leq 1$ for all $a \in B$, $F(\mathbf{1}) = 1$ and $F(K_c) \geq \kappa$ for all $c \in C$.
Consider the convex hull of the set $\{K_c : c \in C\}$:

$$G = \{\sum_{i=1}^{i=m} \alpha_i K_{c_i} : c_i \in C,\, 0 \leq \alpha_i \leq 1,\, \sum \alpha_i = 1\}$$

Lemma. For every $g \in G$, $||g|| \geq \kappa$.

Proof.
First consider rational coefficients α_i: for each $i \leq m$, $\alpha_i = l_i/n$ with $\sum_i l_i = n$, and $g(x) = f(x)/n$ where $f = \sum_i^m l_i \cdot K_{c_i}$. Consider the sequence s in C of length n where each c_i is repeated l_i times. By definition of κ there are k terms of s with nonempty intersection such that $k/n \geq \kappa$. Let x be a point in the intersection; it follows that $f(x) \geq k$. Hence $g(x)) \geq k/n \geq \kappa$. For arbitrary α_i let $\varepsilon > 0$. There are rational approximations β_i of the α_i such that $\kappa \leq ||\sum \beta_i K_{c_i}|| \leq ||\sum \alpha_i K_{c_i}|| + \varepsilon$. Hence $||g|| \geq \kappa - \varepsilon$, and so $||g|| \geq \kappa$.

Let $Q = \{\alpha(g - \kappa) + f : g \in G,\ \alpha \geq 0,\ f \geq \mathbf{0}\}$.
The set Q contains all K_b, $b \in B$, (because $K_b \geq \mathbf{0}$) and is convex: if f and g are in Q and $\alpha + \beta = 1$ $(\alpha,\ \beta > 0)$ then $\alpha f + \beta g \in Q$.
Clearly, it suffices to verify this for f and g in G and that is easy.
Let $\delta = 1 - \kappa$, and let U be the open ball $\{h : ||h|| < \delta\}$ of radius δ. Using the vector space convention
$A - B = \{a - b : a \in A,\ b \in B\}$, consider the set
$U - Q = \{h - \alpha(g - \kappa) - f : ||h|| < \delta,\ g \in G,\ \alpha \geq 0,\ f \geq \mathbf{0}\}$.
The set $U - Q$ is convex, and because $\mathbf{0} \in Q$, we have
$U \subset U - Q$, and so for every $v \in V$ there exists a positive number α such that $\alpha v \in U \subset U - Q$. Now define
$p(v) = \inf\{\gamma > 0 : \frac{v}{\gamma} \in U - Q\}$.
If $\gamma = \alpha + \beta$ and $\frac{x}{\alpha}, \frac{y}{\beta} \in U - Q$ then $\frac{x+y}{\gamma} = \frac{\alpha}{\gamma}(\frac{x}{\alpha}) + \frac{\beta}{\gamma}(\frac{y}{\beta}) \in U - Q$ by convexity, and so $p(x + y) \leq p(x) + p(y)$.
Clearly, $p(\alpha v) = \alpha p(v)$ for all $\alpha \geq 0$.

Finally, let $\gamma \leq 1$; we show that $1/\gamma \notin U - Q$, hence $p(\mathbf{1}) \geq 1$. If $1/\gamma \in U - Q$ then $1/\gamma = h - f$ (and $h = f + 1/\gamma$) where $h \in U$ and $f \in Q$. Since $f \in Q$, there exists, by the Lemma, some $x \in S$ such that $f(x) \geq 0$, and so $f(x) + 1/\gamma \geq 1/\gamma \geq 1$. But $h(x) < \delta \leq 1$.
Now we apply the Hahn-Banach Theorem to this function p. Note that for all $f \in Q$, $-f \in U - Q$ and hence $p(-f) \leq 1$.
There exists a linear functional F such that $F(\mathbf{1}) = 1$ and $F(x) \leq p(x)$ for all $x \in V$. If $f \in Q$, then $F(-f) \leq p(-f) \leq 1$ and so $F(f) \geq -1$. As this is true for all $f \in Q$ and Q is closed under multiples by all $\alpha \geq 0$, it must be the case that $F(f) \geq 0$ for all $f \in Q$. In particular, $F(K_b) \geq 0$ for all $b \in B$.
Also, if $g \in G$, then $g - \kappa \in Q$ and hence $F(g - \kappa) \geq 0$, i.e. $F(g) \geq \kappa$. Consequently, $F(K_c) \geq \kappa$ for all $c \in C$.
When we let $m(b) = F(K_b)$ for all $b \in B$, m is a finitely additive measure on B with $m(c) \geq \kappa$ for all $c \in C$.

We complete the proof by showing that Kelley's condition applies:

Theorem. Let B be a Boolean algebra that has a graded σ-bounded cc fragmentation $\{C_n\}$. Then for every n, C_n has a positive intersection number.

To prove the theorem, we adapt the Kalton-Roberts proof, an ingenious combinatorial argument that converts finite bounds for the size of antichains into positive intersection numbers.

Lemma. Let M and P be finite sets with $|M| = m$ and $|P| = p \leq m$, and let k, $3 \leq k \leq p$ be an integer such that $p/k \geq 15 \cdot m/p$. Then there exists an indexed family $\{A_i : i \in M\}$ such that each A_i is a three point subset of P and such that for every $I \subset M$ with $|I| \leq k$,

$$|\bigcup_{i \in I} A_i| > |I|.$$

It follows that for every $I \subset M$ with $|I| \leq k$ there exists a one-to-one choice function f_I on $\{A_i : i \in I\}$.

The last statement of the lemma follows by **Hall's "Marriage Theorem":**

A family $\{A_1, ..., A_n\}$ of finite sets has a set of distinct representatives if and only if $|\bigcup_{i \in I} A_i| \geq |I|$ for every $I \subset \{1, ..., n\}$.

(For a proof of Hall's Theorem, see Appendix.)

The proof of the lemma uses a counting argument:

Proof. Consider the families $\{A_i : i \in M\}$ of three point subsets of P. Let us call such a family *bad* if $|\bigcup_{i\in I} A_i| \le |I|$ for some $I \subset M$, $|I| \le k$. If a family is bad then for some n, $3 \le n \le k$, there exist sets $I \subset M$ and $J \subset P$, $|I| = |J| = n$ such that $A_i \subset J$ for every $i \in I$.

There are $\binom{p}{3}$ three-point subsets of P and $\binom{n}{3}$ three-point subsets of J. Of the $\binom{p}{3}^n$ families $\{A_i\}_{i\in I}$ with domain I, $\binom{n}{3}^n$ are such that $\bigcup_{i\in I} A_i \subset J$. The ratio of such families (for $3 \le n \le p$) is $\left(\binom{n}{3}/\binom{p}{3}\right)^n \le (n^3/p^3)^n = n^{3n}/p^{3n}$ because $\binom{n}{3}/\binom{p}{3} \le n^3/p^3$.
Because there are $\binom{m}{n}$ subsets $I \subset M$ of size n and $\binom{p}{n}$ subsets $J \subset P$ of size n, the probability that a family $\{A_i\}_{i\in M}$ is bad is at most

$$\Pi = \sum_{n=3}^{n=k} \binom{m}{n}\binom{p}{n}\frac{n^{3n}}{p^{3n}}.$$

The Kalton-Roberts Proof

We have $\binom{m}{n} \cdot \binom{p}{n} \cdot n^{3n}/p^{3n} \leq m^n/n! \cdot p^n/n! \cdot n^{3n}/p^{3n}$. Using $e^x > x^n/n!$ we get $e^n n! > n^n$, hence $1/n! < e^n/n^n$, and so

$$\frac{m^n}{n!} \cdot \frac{p^n}{n!} \cdot \frac{n^{3n}}{p^{3n}} < \frac{e^{2n} n^n m^n}{p^{2n}} = (e^2 \cdot \frac{n}{p} \cdot \frac{m}{p})^n.$$

For $n \leq k$ we have $e^2 \cdot n/p \cdot m/p \leq e^2 \cdot k/p \cdot m/p \leq e^2/15 < 1/2$ because we assumed $p/k \geq 15m/p$ and because $2e^2 < 15$.
Therefore

$$\Pi < \sum_{n=3}^{n=k} \left(\frac{1}{2}\right)^n < 1.$$

Consequently, there exists a family $\{A_i : i \in M\}$ that is not bad, and so $|\bigcup_{i \in I}| > |I|$ for every $I \subset M$ of size $\leq k$.

The Kalton-Roberts Proof

We shall now apply the Kalton-Roberts method:

Proof of Theorem. Let $\{C_n\}$ be a graded σ-bounded cc fragmentation of a Boolean algebra B, and let us fix an integer n. We prove that the intersection number of each C_n is positive, namely $\geq 1/(30K^2)$ where $K = K_{n+2}$ is the maximal size of an antichain in C_{n+2}.

We show that for every $m \geq 100K^2$, and every sequence $\{c_1, ..., c_m\}$ in C_n there exists some $J \subset m$ of size $\geq m/(30K^2)$ such that $\bigcap_{i\in J} c_i$ is nonempty.

Let $M = \{1, ..., m\}$ with $m \geq 100K^2$ and let $c_1, ..., c_m \in C_n$. For each $I \subset M$, let

$$b_I = \bigcap\{c_i : i \in I\} \cap \bigcap\{-c_i : i \notin I\}.$$

The sets b_I are pairwise disjoint (some may be empty) and $\bigcup\{b_I : I \subset M\} = \mathbf{1}$. Note that for each $i \in M$, $\bigcup\{b_I : i \in I\} = c_i$. We shall find a sufficiently large set $J \subset M$ with nonempty b_J.

We shall apply the Lemma. First let $k \geq 3$ be the largest k such that $k/m < 1/(30K^2)$ (there is such because $3/m \leq 3/(100K^2)$. We have $k < m$ and $(k+1)/m \geq 1/(30K^2)$. Then let p be the largest $p \geq k$ such that $p/m < 1/K$ (there is such because $k/m < 1/K$.)

We verify the assumption of the lemma, $p/k \geq 15m/p$ (using $p/(p+1) \geq 3/4$):

$$\frac{p}{k} = \frac{p}{p+1} \cdot \frac{p+1}{m} \cdot \frac{m}{k} \geq \frac{3}{4} \cdot \frac{1}{K} \cdot 30K^2 \geq 20K$$

and

$$15\frac{m}{p} \leq 15 \cdot \frac{p+1}{p} \cdot \frac{m}{p+1} \leq 15 \cdot \frac{4}{3} \cdot K = 20K.$$

The Kalton-Roberts Proof

Now we apply the Lemma: Let $P = \{1, ..., p\}$. There exist three point sets $A_i \subset P$, $i \in M$, and one-to-one functions f_I on all $I \subset M$ of size $\leq k$ with $f_I(i) \in A_i$ for all $i \in I$.

We shall prove that there exists a $J \subset M$ of size $\geq k+1$ (and hence $\geq m/(30K^2)$) such that b_J is nonempty. By contradiction, assume that there is no such J. Then

$\bigcup\{b_I : |I| \leq k\} = \mathbf{1}$ and for each $i \in M$, $c_i = \bigcup\{b_I : |I| \leq k \text{ and } i \in I\}$.

For each $i \in M$ and $j \in P$ let

$$a_{ij} = \bigcup\{b_I : |I| \leq k,\ i \in I \text{ and } f_I(i) = j\}.$$

Note that for each $i \in M$, $c_i = a_{i,j_1} \cup a_{i,j_2} \cup a_{i,j_3}$ where $A_i = \{j_1, j_2, j_3\}$.

The Kalton-Roberts Proof

Let $j \in P$. We claim that the a_{ij}, $i \in M$, are pairwise disjoint: If $a_{i_1,j} \cap a_{i_2,j}$ is nonempty, then because the b_I are pairwise disjoint there is some I such that $i_1 \in I$ and $i_2 \in I$, and because $f_I(i_1) = j = f_I(i_2)$ and f_I is one-to-one, we have $i_1 = i_2$. Hence the a_{ij}, $i \in M$, are pairwise disjoint, and so only at most K of them belong to C_{n+2}.

Consequently, at most $p \cdot K$ of the a_{ij} belong to C_{n+2} and because $pK < m$, there exists an i such that $a_{ij} \notin C_{n+2}$ for all (three) $j \in A_i$.

But then $c_i = a_{i,j_1} \cup a_{i,j_2} \cup a_{i,j_3} \notin C_n$ because the fragmentation is graded. This contradicts the assumption that $c_i \in C_n$.